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On the global evolution problem in 2 + 1 gravity

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Abstract

Existence of global constant mean curvature (CMC) foliations of constant curvature 3-dimensional maximal globally hyperbolic Lorentzian manifolds, containing a constant mean curvature hypersurface with genus$(\Sigma) > 1$, is proved. Constant curvature 3-dimensional Lorentzian manifolds can be viewed as solutions to the 2 + 1 vacuum Einstein equations with a cosmological constant. The proof is based on the reduction of the corresponding Hamiltonian system in CMC gauge to a time-dependent Hamiltonian system on the cotangent bundle of Teichmüller space. Estimates of the Dirichlet energy of the induced metric play an essential role in the proof.

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1. Introduction

Lichnerowicz [2] used the conformal transformation properties of the scalar curvature to write the constraint equations on Cauchy data for the Einstein equations as a semilinear elliptic system. This important insight together with the fact that in the case of constant mean curvature (CMC) data the Hamiltonian and the momentum constraint equations decouple, leads to the conformal method for solving the constraint equations and to the conformal method of reduction of the Einstein equations in CMC gauge.

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The Einstein equations in $3+1$ dimensions on a spacetime of topology $\Sigma \times I$ for some interval I reduce, using the conformal method (ignoring technical difficulties) to a time-dependent Hamiltonian system on $T^*\mathcal{S}(\Sigma)$, where $\mathcal{S}(\Sigma)$ is the "conformal superspace" of Σ, i.e. the space of Riemannian metrics on Σ modulo conformal rescalings and diffeomorphisms. In the $2+1$-dimensional case with Σ a compact oriented surface, the above ideas can be carried out in full detail [4].

In this paper we address the evolution problem for $2+1$-dimensional vacuum gravity with a cosmological constant in CMC gauge. We prove that all $2+1$-dimensional vacuum spacetimes $(\Sigma \times I, \bar{g})$ satisfying the vacuum Einstein equations with cosmological constant,

$$\bar{R}_{ab} = \Lambda \bar{g}_{ab},$$

which contain a CMC hypersurface are globally foliated by CMC hypersurfaces. In this case Eq. (31) below corresponds to the Lichnerowicz equation.

Vacuum $2+1$-dimensional spacetimes with a cosmological constant are 3-dimensional Lorentzian spaceforms. The problem of classifying the 3-dimensional maximal globally hyperbolic Lorentzian spaces of constant curvature, and of topology $\Sigma \times \mathbb{R}$ was solved by Mess in [3] for $\Lambda \leq 0$. The question of existence of a global CMC foliation was left open in this work.

1.1. Overview

This paper is organized as follows. In Section 2 we review some facts from Teichmüller theory and define the Dirichlet energy $\widehat{E}$. Here $\widehat{E}$ is the form of the Dirichlet energy with fixed target space introduced by Tromba [5]. The fact that this is a proper function on Teichmüller space is a key ingredient in our argument. In Section 3 we introduce the $2+1$ Einstein evolution equations in CMC gauge and derive the Calabi–Simons identity (20). Since we are dealing with constant curvature spacetimes the Riccati equation can be explicitly solved. This is done in Section 3.1, where for $\Lambda \geq 0$ we use this to study the asymptotics of the mean curvature of the Gauss foliation. In Section 4 we state and prove the main results, Theorem 6 (global existence) and Corollary 7 (global CMC foliations).

The conformal constraint equations are considered in Section 4.1. In Section 4.2, some estimates for the area and the Dirichlet energy are derived. An application of the Calabi–Simons identities, Lemma 3, for the second fundamental form together with the maximum principle yields pointwise bounds on the second fundamental form K in terms of the mean curvature τ. This bound together with the Einstein evolution equations allows one to derive a differential inequality for the time development of the Dirichlet energy $\widehat{E}(g(\tau))$ of the metric $g(\tau)$ which shows that $\widehat{E}(g(\tau))$ is finite for all τ allowed by the constraint equations. Finally, in Section 4.3 the proof of the main theorem is carried out, using the bound on the Dirichlet energy derived in Section 4.2.

2. Teichmüller space

In this section we present some background material on Teichmüller space. The main reference is the book by Tromba [5]. Fix a compact oriented 2-manifold Σ of genus > 1. Let $\mathcal{M}$ denote the space of C^∞ metrics on Σ, let $\mathcal{M}_{-1} \subset \mathcal{M}$ denote the space of metrics of constant scalar curvature -1, let $\mathcal{D}_0$ denote the group of C^∞ diffeomorphisms of Σ isotopic to the identity and let $\mathcal{P}$ be the group of positive C^∞ functions on Σ.

The scalar curvature function obeys the following transformation rule in 2 dimensions:

$$R[\mathrm{e}^{2\lambda}g] = \mathrm{e}^{-2\lambda}(-2\Delta_g\lambda + R[g]). \tag{1}$$

The equation

$$\Delta_g\lambda = \tfrac{1}{2}(R[g] + \mathrm{e}^{2\lambda})$$

has a unique solution in case genus$(\Sigma) > 1$ and therefore we may construct a metric $h = \mathrm{e}^{2\lambda}g \in \mathcal{M}_{-1}$ conformal to g and with constant scalar curvature -1. $\mathcal{P}$ acts on $\mathcal{M}$ by conformal rescaling and $\mathcal{M}_{-1}$ is a global slice for the action of $\mathcal{P}$. Thus

$$\mathcal{M}_{-1} = \mathcal{M}/\mathcal{P}.$$

Any C^∞ symmetric 2-tensor k can be decomposed uniquely as

$$k = k_{\mathrm{TT}} + fg + L_g(Y), \tag{2}$$

where k_{TT} is transverse traceless, i.e. $\mathrm{tr}_g k_{\mathrm{TT}} = 0$, $\mathrm{div}_g k_{\mathrm{TT}} = 0$, $f \in C^\infty(\Sigma)$ and $L_g(Y)$ is the conformal Killing form for the vector field Y, i.e. the trace free part of $\mathcal{L}_Y g$,

$$L_g(Y) = \mathcal{L}_Y g - \tfrac{1}{2}\mathrm{tr}_g(\mathcal{L}_Y g)g.$$

The decomposition (2) is L^2 orthogonal. In 2 dimensions div acting on traceless symmetric 2-tensors is elliptic and by Riemann–Roch the dimension of the kernel has dimension $6\,\mathrm{genus}(\Sigma) - 6$ for genus$(\Sigma) > 1$. Further in 2 dimensions, transverse traceless is a conformally invariant property. We will use the notation $S^2_{\mathrm{TT}}(T^*\Sigma, h)$ for the space of TT-tensors w.r.t. h.

Let $k \in T_h\mathcal{M}_{-1}$. Then k is of the form $k_{\mathrm{TT}} + \mathcal{L}_X h$ for some X. The TT-tensors provide a local slice for the action of D_0 on $\mathcal{M}_{-1}$. The action of $\mathcal{D}_0$ is proper so Teichmüller space $\mathcal{T}(\Sigma)$ defined by

$$\mathcal{T}(\Sigma) = \mathcal{M}_{-1}/\mathcal{D}_0$$

is a manifold of dimension $6\,\mathrm{genus}(\Sigma) - 6$.

$\mathcal{M}$ and $\mathcal{M}_{-1}$ are Riemannian manifolds w.r.t. the L^2-metric defined for $h, k \in T_g\mathcal{M}$ by

$$\langle\langle h, k\rangle\rangle_g = \int_\Sigma \langle h, k\rangle_g \sqrt{g}\,\mathrm{d}^2x$$

and $\frac{1}{2}\langle\langle\cdot,\cdot\rangle\rangle$ restricted to $\mathcal{M}_{-1}$ induces a Riemannian structure $\langle\cdot,\cdot\rangle_{WP}$ on $\mathcal{T}(\Sigma)$, the Weil–Peterson metric.

We need some basic facts about harmonic mappings. Let G be a fixed element in $\mathcal{M}_{-1}$ and for $g \in \mathcal{M}_{-1}$, $S : \Sigma \to \Sigma$, let

$$e(S, g) = \frac{1}{2} g^{ij} \frac{\partial S^\alpha}{\partial x^i} \frac{\partial S^\beta}{\partial x^j} G_{\alpha\beta} = \frac{1}{2} \langle \nabla S^\alpha, \nabla S^\beta \rangle_g G_{\alpha\beta}$$

and let $E(S, g)$ be given by

$$E(S, g) = \int_\Sigma e(S, g) \sqrt{g}\, \mathrm{d}^2 x. \tag{3}$$

There is a unique map homotopic to the identity which minimizes $E(S, g)$. This map, which we denote by $S(g)$, is a harmonic map $(\Sigma, g) \to (\Sigma, G)$ and it can be proved that $S(g) \in \mathcal{D}_0$.

Let $\sigma : \mathcal{M}_{-1} \to \mathcal{M}_{-1}$ be defined by

$$\sigma(g) = S(g)_* g. \tag{4}$$

Since $S(g) \in \mathcal{D}_0$ the pushforward is well defined. By the uniqueness of $S(g)$ we have $\sigma(f^* g) = \sigma(g)$ for any $f \in \mathcal{D}_0$. Therefore σ induces a map

$$\sigma : \mathcal{T}(\Sigma) \to \mathcal{M}_{-1}, \tag{5}$$

which is a global slice for the action of $\mathcal{D}_0$, see [5, Section 3.4]. By a slight abuse of notation we will say that $h \in \sigma$ if $h \in \sigma(\mathcal{T}(\Sigma)) \subset \mathcal{M}_{-1}$. Given $h \in \mathcal{M}_{-1}$ or $g \in \mathcal{M}$, we denote the corresponding classes in $\mathcal{T}(\Sigma)$ by $[h]$ and $[g]$.

We summarize the relevant facts in:

Proposition 1 [5].

(i) *$\mathcal{M}_{-1} = \mathcal{M}/\mathcal{P}$ and $\mathcal{M}_{-1}$ is a global slice for the action of $\mathcal{P}$.*

(ii) *$\mathcal{T}(\Sigma) = \mathcal{M}_{-1}/\mathcal{D}_0$ and there is a global slice for the action of $\mathcal{D}_0$ given by the map $\mathcal{T}(\Sigma) \to \mathcal{M}_{-1}$ induced by the diffeomorphism invariant map $\mathcal{M}_{-1} \to \mathcal{M}_{-1}$ given by $\sigma(g) = S(g)_* g$, with $S(g) \in \mathcal{D}_0$ the harmonic minimizer of* (3). *$\mathcal{T}(\Sigma)$ is a C^∞ manifold of dimension* $\dim \mathcal{T}(\Sigma) = 6\,\mathrm{genus}(\Sigma) - 6$ *diffeomorphic to* $\mathbb{R}^{6\,\mathrm{genus}(\Sigma)-6}$.

(iii) *The weak Riemannian metric on $\mathcal{M}_{-1}$ induces a metric $\langle \cdot, \cdot \rangle_{WP}$ on $\mathcal{T}(\Sigma)$, the Weil–Peterson metric. $\mathcal{T}(\Sigma)$ is geodesically convex w.r.t. the Weil–Peterson metric but not complete.*

We define the Dirichlet energy of $g \in \mathcal{M}_{-1}$ to be

$$E(g) = E(S(g), g). \tag{6}$$

$E(g)$ turns out to be conformally invariant and diffeomorphism invariant. By the conformal invariance, $E(g)$ extends to a function on $\mathcal{M}$ which we denote by $\widehat{E}$. Further, by the diffeomorphism invariance of E it defines a function $\widetilde{E}$ on $\mathcal{T}(\Sigma)$ which we again call the Dirichlet energy.

Proposition 2 [5].

(i) *The Dirichlet energy* $\widetilde{E}$ *is a proper function on* $\mathcal{T}(\Sigma)$.

(ii) *Let* $g \in \mathcal{M}$ *and let* $S(g) \in \mathcal{D}_0$ *be the harmonic map as in* (6). *For* $h \in T_g(\mathcal{M})$, *let* $\tilde{h}^{\sharp}$ *denote the* (1, 1)*-version of the trace free part of* h. *Then*

$$D\widehat{E}(g)h = -\frac{1}{2}\int_{\Sigma} \langle \tilde{h}^{\sharp}\nabla S^l, \nabla S^l\rangle_g \sqrt{g}\,\mathrm{d}^2x. \tag{7}$$

3. Vacuum $2+1$ gravity

Assume that Σ is a compact and orientable 2-dimensional manifold and let $(\Sigma \times I, \bar{g})$ be a maximal globally hyperbolic $2+1$ spacetime, which solves the Einstein vacuum equations with cosmological constant Λ. We will denote the covariant derivative and curvature tensor defined w.r.t. $\bar{g}$ by $\bar{\nabla}$, $\bar{R}$, respectively. The field equations are

$$\bar{R}_{ab} = \Lambda \bar{g}_{ab}. \tag{8}$$

Note that Eq. (8) due to the 3-dimensionality of $\Sigma \times I$ is equivalent to

$$\bar{R}_{abcd} = \tfrac{1}{2}\Lambda(\bar{g}_{ac}\bar{g}_{bd} - \bar{g}_{ad}\bar{g}_{bc}),$$

i.e. $(\Sigma \times I, \bar{g})$ is a Lorentzian spaceform of sectional curvature $\frac{1}{2}\Lambda$.

We assume that the constant time hypersurfaces $\Sigma_t = \Sigma \times \{t\}$ are spacelike with normal T and denote the induced metric by g. Define the lapse function N and shift vectorfield X by

$$N = -\langle \partial_t, \mathrm{T}\rangle, \qquad X = X^a\partial_{x^a} = \partial_t - N\cdot \mathrm{T}.$$

Then we can write $\bar{g}$ in the form

$$\bar{g} = -N^2\,\mathrm{d}t \otimes \mathrm{d}t + g_{ab}(\mathrm{d}x^a + X^a\,\mathrm{d}t) \otimes (\mathrm{d}x^b + X^b\,\mathrm{d}t). \tag{9}$$

Let K denote the second fundamental form, i.e.

$$K_{ab} = -\langle \bar{\nabla}_a \mathrm{T}, e_b\rangle = \langle \mathrm{T}, \bar{\nabla}_a e_b\rangle$$

and let

$$\pi^{ab} = K_c^{\ c} g^{ab} - K^{ab}, \tag{10}$$

then $\pi'^{ab} = \sqrt{g}\pi^{ab}$ is the canonically conjugate variable to g in the Hamiltonian formulation of Einsteins equations. Note that in [4] the notation π^{ab} rather than π'^{ab} was used.

In the $2+1$ case with cosmological constant Λ we have from the Gauss and Gauss–Codazzi equations

$$(\mathrm{tr}_g\pi)^2 - |\pi|_g^2 + R = \Lambda, \tag{11}$$

$$\nabla_a\pi^{ab} = 0. \tag{12}$$

Using the second variation equations and the definition of N, X one arrives after a bit of calculation at the equations of motion which in terms of (g, π) are

$$\partial_t g_{ab} = 2N(\pi_{ab} - (\mathrm{tr}_g \pi) g_{ab}) + (\mathcal{L}_X g)_{ab}, \tag{13}$$

$$\begin{aligned} \partial_t \pi_{ab} = {} & \nabla_a \nabla_b N - \Delta N g_{ab} - N \frac{\Lambda}{2} g_{ab} \\ & + N[|\pi|_g^2 g_{ab} - (\mathrm{tr}_g \pi)^2 g_{ab} + \pi_{ac} \pi^c{}_b] + (\mathcal{L}_X \pi)_{ab}. \end{aligned} \tag{14}$$

Let $\tau = \mathrm{tr}_g \pi$. We will in the following consider only the case of CMC data, with $\mathrm{d}\tau = 0$. The lapse function corresponding to a CMC slicing satisfies

$$-\Delta N + (|\pi|_g^2 - \Lambda) N = 1. \tag{15}$$

By (2) and the momentum constraint equation (12), π satisfies

$$\pi = \tfrac{1}{2} \tau g + \pi_{\mathrm{TT}}, \qquad |\pi|_g^2 = \tfrac{1}{2} \tau^2 + |\pi_{\mathrm{TT}}|_g^2, \tag{16}$$

in the CMC case.

In case $\Sigma \cong S^2$, then $\mathcal{T}(\Sigma)$ is zero-dimensional and there are no nontrivial TT-tensors. It follows from this and the constraint equations (11) and (12) that for $\Sigma \cong S^2$, the $2+1$ vacuum Einstein equations have solutions only in case $\Lambda > 0$, $\tau^2 < 2\Lambda$ and the solution in this case is given by

$$g_{ab}(\tau) = g_{ab}(\tau_0) \frac{\tau_0^2 - 2\Lambda}{\tau^2 - 2\Lambda}. \tag{17}$$

This is the $2+1$-dimensional deSitter universe.

If $\Sigma \cong T^2$, any TT-tensor is covariant constant and the space of TT-tensors is 2-dimensional. The equations of motion can be explicitly solved.

For genus$(\Sigma) > 1$ we do not have an explicit solution of the equations of motion except in the trivial case $\pi_{\mathrm{TT}} = 0$. In this case, the solutions of the field equations can be described as follows. If genus$(\Sigma) > 1$ and $\Lambda \geq 0$, the constraint equations imply $\tau^2 > 2\Lambda$ and we may therefore assume $\tau_0 > \sqrt{2\Lambda}$. The evolution of trivial data is given by (17). We see that in the case when $\Lambda \geq 0$, the trivial solutions undergo an infinite expansion as $\tau \searrow \sqrt{2\Lambda}$, collapse to a singularity as $\tau \to \infty$, while, when $\Lambda < 0$, τ runs from $-\infty$ to $+\infty$ and we have a "big bang" and a "big crunch". In case $\Lambda = 0$, the trivial solutions correspond to quotients of the interior of a light-cone in the $2+1$-dimensional Minkowski space, while in case $\Lambda < 0$, the trivial solutions correspond to quotients of a maximal globally hyperbolic subset of the $2+1$-dimensional anti-deSitter space.

Lemma 3. *The following identities hold for* $K \in S^2_{\mathrm{TT}}(T^*\Sigma, g)$.

$$\nabla_c K_{ab} - \nabla_b K_{ac} = 0, \tag{18}$$

$$\nabla^c \nabla_c K_{ab} = R K_{ab}, \tag{19}$$

$$\tfrac{1}{2} \Delta |K|_g^2 = |\nabla K|_g^2 + R |K|_g^2. \tag{20}$$

Proof. To prove the identity (18) note that if $K \in S^2_{\mathrm{TT}}(T^*\Sigma, g)$ then in an ON frame

$$K_{11;1} + K_{12;2} = 0, \qquad K_{21;1} + K_{22;2} = 0,$$
$$K_{11;1} + K_{22;1} = 0, \qquad K_{11;2} + K_{22;2} = 0,$$

which after an elementary manipulation gives (18). Let $K \in S^2_{\mathrm{TT}}(T^*\Sigma, g)$. A computation in an ON frame gives

$$K_{ij;kk} = K_{ik;jk}$$
$$K_{ij;kk} = K_{ik;jk} = K_{ik;kj} + R_{fijk}K_{fk} + R_{fkjk}K_{if}$$

By (18) and $K \in S^2_{\mathrm{TT}}(T^*\Sigma, g)$, $K_{ik;kj} = K_{kk;ij} = 0$ and using the fact that in 2 dimensions, the Riemann tensor is of the form

$$R_{abcd} = \tfrac{1}{2}R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

gives (19). Finally, (20) is proved from (19) by a straightforward computation. □

Remark 4. Note that the identities (18) and (19) are special to 2 dimensions. Eq. (20) is one of the Calabi–Simons identities, analogs of this hold for Codazzi tensors in higher dimensions, and for higher covariant derivatives of K, see [1] and references therein. The above identity may also be proved by differentiating the 2-dimensional identity $R_{ab} - \frac{1}{2}Rg_{ab} = 0$ in the direction of a TT tensor.

The Calabi–Simons identity enables us to apply the maximum principle to get pointwise estimates for π_{TT} and N.

Lemma 5. *Assume* genus$(\Sigma) > 1$. *Let* (g_{ab}, π^{ab}) *be CMC data with mean curvature* τ, *for* $2+1$ *vacuum GR with cosmological constant* Λ *and let* N *be the lapse function. Then* $\tau^2/2 > \Lambda$ *and*

$$|\pi_{\mathrm{TT}}|_g^2 \le \frac{\tau^2 - 2\Lambda}{2}, \tag{21}$$

$$\frac{1}{\tau^2 - 2\Lambda} \le N \le \frac{2}{\tau^2 - 2\Lambda}. \tag{22}$$

Proof. Let π_{TT} be the TT part of π and apply Lemma 3. The maximum principle and (20) implies that at a maximum of $|\pi_{\mathrm{TT}}|_g^2$,

$$0 \ge R.$$

Using (16) and the Hamiltonian constraint (11) proves (21). It follows that $\tau^2 \ge 2\Lambda$, but equality implies $R = 0$ which is ruled out by genus$(\Sigma) > 1$. Similarly, applying the maximum principle using (15) proves (22). □

Note that in the case when $\Lambda \ge 0$, the range of mean curvatures τ is limited by $2\Lambda < \tau^2 < \infty$.

3.1. Gauss coordinates

Before studying the evolution in the CMC gauge, it is instructive to consider the geometry of the spacetime in Gauss coordinates. Due to the fact that $(\Sigma \times I, \bar{g})$ has constant curvature, we can integrate the Riccati equation.

Let $f : \Sigma \times \mathbb{R}^+ \to \Sigma \times I$ be the normal exponential map, defined by $f_s(x) = \exp_x(sT)$ and let $F(s)$ be defined by

$$F(s)X = f_{s*}X,$$

where f_{s*} denotes push forward under f_s. Then F satisfies the Jacobi equations

$$\ddot{F} + \bar{R}_T F = 0, \tag{23}$$

where $\bar{R}_T$ is defined by $\bar{R}_T X = \bar{R}(X, T)T$, i.e. in the case of a 2+1-dimensional spacetime of constant curvature $\frac{1}{2}\Lambda$, we have

$$\bar{R}_T = -\tfrac{1}{2}\Lambda I.$$

Let S be the Weingarten map given by $SX = -\nabla_X T$ where T is the timelike normal to $f_s(\Sigma)$. Then $S = -\dot{F}F^{-1}$ and S satisfies the Riccati equation

$$\dot{S} = S^2 + \bar{R}_T. \tag{24}$$

In the constant curvature case, Eq. (23) has with $F(0) = I$, $\dot{F}(0) = -S_0$, the solution

$$F(s) = I - sS_0 \quad \text{for } \Lambda = 0, \tag{25}$$

$$F(s) = I \cosh\left(\sqrt{\frac{\Lambda}{2}}s\right) - \sqrt{\frac{2}{\Lambda}}S_0 \sinh\left(\sqrt{\frac{\Lambda}{2}}s\right), \quad \text{for } \Lambda > 0, \tag{26}$$

$$F(s) = I \cos\left(\sqrt{\frac{|\Lambda|}{2}}s\right) - \sqrt{\frac{2}{|\Lambda|}}S_0 \sin\left(\sqrt{\frac{|\Lambda|}{2}}s\right), \quad \text{for } \Lambda < 0. \tag{27}$$

Let genus$(\Sigma) > 1$ and assume that (g_0, S_0) are the metric and Weingarten map of a hypersurface Σ_{τ_0} with CMC τ_0 in a $2+1$ vacuum spacetime with cosmological constant Λ. Recall that S is just the $(1, 1)$-form of the second fundamental form K. It follows from (21) that in case $\Lambda \geq 0$, after a choice of time orientation,

$$S_0 \leq \frac{-\tau_0 + \sqrt{\tau_0^2 - 2\Lambda}}{2} I < 0.$$

In case $\Lambda \geq 0$ we find from (25) and (26) that the solution to the Jacobi equation (23) exists and is nondegenerate for all $s > 0$ and hence Σ_{τ_0} has no focal points in the expanding direction. A computation shows that

$$\lim_{s\to\infty} \operatorname{tr} S(s) = -\sqrt{2\Lambda}.$$

Finally, we note that in case $\Lambda \geq 0$ we have causal geodesic completeness in the expanding direction. To see this, use that by global hyperbolicity and the fact that the time variable

s for the Gauss foliation is proper time, any point in $\Sigma \times I$ which is in the expanding direction w.r.t. Σ_{τ_0} is on one of the leaves in the Gauss foliation and hence cannot be an endpoint for an inextendible causal geodesic.

4. Global existence

We now state the main theorem.

Theorem 6. *Fix a compact oriented 2-manifold Σ of genus > 1. Let $(\Sigma \times I, \bar{g})$ be a globally hyperbolic spacetime solving the vacuum Einstein equations* (8) *with cosmological constant Λ, which is the maximal globally hyperbolic development of CMC data (g, π) on Σ with mean curvature τ_0. If $\Lambda \geq 0$, assume that $\tau_0 > \sqrt{2\Lambda}$. Then the following is true:*

(i) *The Einstein evolution equations with CMC time gauge and spatial gauge given by the slice σ of Proposition* 1*(ii) has solution for all τ allowed by the constraint equations, i.e. for*

$$\begin{aligned} \sqrt{2\Lambda} < \tau < \infty, \quad & \Lambda \geq 0, \\ -\infty < \tau < \infty, \quad & \Lambda < 0. \end{aligned}$$

(ii) *The area* $\mathrm{Area}(\Sigma, g(\tau))$ *of Σ w.r.t. the induced metric $g(\tau)$ at mean curvature time τ satisfies* $\mathrm{Area}(\Sigma, g(\tau)) \to 0$ *as $\tau \to \pm\infty$ and in the case $\Lambda \geq 0$,* $\mathrm{Area}(\Sigma, g(\tau)) \to \infty$ *as $\tau \searrow \sqrt{2\Lambda}$.*

(iii) *In case $\Lambda \geq 0$, for $\sqrt{2\Lambda} < \tau < \tau_0$,*

$$\widehat{E}(g(\tau)) \leq \widehat{E}(g(\tau_0)) \left(\frac{\tau_0 + \sqrt{\tau_0^2 - 2\Lambda}}{\tau + \sqrt{\tau^2 - 2\Lambda}} \right)^{\sqrt{2}}.$$

In particular, for $\Lambda > 0$, the Dirichlet energy $\widehat{E}(g(\tau))$ is bounded for the evolution in the expanding direction $\tau \searrow \sqrt{2\Lambda}$ and the class of $g(\tau)$ stays in a compact subset of $\mathcal{T}(\Sigma)$.

Given the global existence for the evolution in CMC time we are now able to prove that the spacetime is globally foliated by CMC hypersurfaces.

Corollary 7. *Let $(\Sigma \times I, \bar{g})$ be as in Theorem* 6*. Then $(\Sigma \times I, \bar{g})$ is globally foliated by CMC hypersurfaces.*

Proof. Let Σ_τ denote a CMC hypersurface with mean curvature τ as constructed in Theorem 6. First we consider the case when $\tau \to \pm\infty$, i.e. the collapsing direction. By a choice of time orientation it is sufficient to consider the case $\tau \nearrow \infty$. If we can show that the focal distance along future directed normal geodesics to the CMC hypersurface Σ_τ tends to 0 as $\tau \nearrow \infty$ then it follows by global hyperbolicity that the CMC foliation constructed in Theorem 6 exhausts the spacetime in the collapsing direction. Let F be defined as in

Section 3.1 by solving the Jacobi equation w.r.t. future directed normal geodesics to Σ_τ for some large τ. Focal points correspond precisely to zero eigenvalues of F. Using the explicit form of F given by (25)–(27) and the fact that at least one of the eigenvalues of the Weingarten map S of Σ_τ is larger than $\frac{1}{2}\tau_0$, one shows easily that the focal distance tends to zero as $\tau_0 \nearrow \infty$.

Next we consider the case $\Lambda \geq 0$ and $\tau \searrow \sqrt{2\Lambda}$. Fix some τ_0 satisfying the conditions of Theorem 6(i). Using (22) and the definition of the lapse function we see that the Lorentz distance from Σ_τ to any point in the past of Σ_{τ_0} will decrease to zero before τ reaches $\sqrt{2\Lambda}$. Therefore the CMC foliation for $\sqrt{2\Lambda} < \tau < \tau_0$ exhausts the past of Σ_{τ_0}. □

Remark 8. In the case $\Lambda \leq 0$, the case when $\tau \to \pm\infty$ in the proof of Corollary 7 is covered by a standard argument which shows that in the case of a crushing singularity, the CMC hypersurfaces foliate a neighborhood of the singular boundary if the strong energy condition holds. The basic comparison argument used for the proof of this can easily be adapted to cover the present situation.

4.1. The conformal constraint equations

Here we review the conformal procedure for solving the constraint equations, which is an essential step in the reduction of vacuum 2+1 gravity to a Hamiltonian system on $T^*\mathcal{T}(\Sigma)$, the cotangent bundle of Teichmüller space, see [4].

The natural phase space for relativity is $T^*\mathcal{M}$, the cotangent bundle of the space of metrics. As discussed above the fiber in $T^*\mathcal{M}$ consists of contravariant symmetric 2-tensor densities. For simplicity of notation we will work with tensors here.

Let $\mathcal{C}_{\tau,\sigma}$ denote the space of solutions of the constraint equations (11) and (12) over the slice σ given by (5), such that $(g,\pi) \in \mathcal{C}_{\tau,\sigma}$ if and only if (g,π) solves the constraint equations, $\mathrm{tr}_g\pi = \tau$ and $g = \mathrm{e}^{2\lambda}h$ for $h \in \sigma$. Then $\mathcal{C}_{\tau,\sigma}$ inherits a symplectic structure from the L^2 symplectic structure on $T^*\mathcal{M}$.

We will construct a map

$$\mathcal{Y}_{\tau,\sigma} : T^*\mathcal{T}(\Sigma) \to \mathcal{C}_{\tau,\sigma}.$$

Let $h \in \mathcal{M}_{-1}$ and let $\omega \in S^2_{\mathrm{TT}}(T^*\Sigma, h)$. Then, letting

$$g = \mathrm{e}^{2\lambda}h, \tag{28}$$

we also have $\omega \in S^2_{\mathrm{TT}}(T^*\Sigma, g)$ by the 2 dimensionality of Σ. From this we get that $\pi^{ab}_{\mathrm{TT}} = \mathrm{e}^{-4\lambda}p^{ab}_{TT}$ is in $S^2_{\mathrm{TT}}(T\Sigma, g)$.

We are interested in constructing a solution (g,π) to the constraint equations (11) and (12) with $\mathrm{tr}_g\pi = \tau$ for $\tau \in \mathbb{R}$. Clearly π of the form

$$\pi^{ab} = \pi^{ab}_{\mathrm{TT}} + \tfrac{1}{2}\tau g^{ab} \tag{29}$$

will solve (12) and every solution is of this form. It remains to consider the Hamiltonian constraint (11) which can be written

$$\tfrac{1}{2}\tau^2 - |\pi_{\mathrm{TT}}|^2_g + R[g] = \Lambda.$$

From the conformal transformation property of π_{TT} we get

$$|\pi_{\mathrm{TT}}|_g^2 = \mathrm{e}^{-4\lambda}|p_{\mathrm{TT}}|_h^2 \tag{30}$$

and using Eqs. (28) and (1), we see that (11) takes the form

$$\Delta_h \lambda = \tfrac{1}{2}R[h] + \tfrac{1}{4}\mathrm{e}^{2\lambda}(\tau^2 - 2\Lambda) - \tfrac{1}{2}\mathrm{e}^{-2\lambda}|p_{\mathrm{TT}}|_h^2. \tag{31}$$

Existence and uniqueness of solutions to (31) has been proved, see [4]. We now define the map $\mathcal{Y}_{\tau,\sigma}$. Let $(q, p) \in T^*\mathcal{T}(\Sigma)$ be given. By Proposition 1(ii) we may introduce global coordinates

$$q^\alpha, \quad \alpha = 1, \ldots, 6\,\mathrm{genus}(\Sigma) - 6,$$

on $\mathcal{T}(\Sigma)$. It is natural to view $T^*\mathcal{T}(\Sigma)$ as the pullback under σ of the TT part of the cotangent bundle of $\mathcal{M}_{-1}$. The space $\mathcal{M}_{-1}$ is a submanifold of $S^2(T^*\Sigma)$ and therefore $T^*_{\mathrm{TT}}\mathcal{M}_{-1}$ in a natural way consists of contravariant symmetric TT tensor densities. We write a general element as

$$p'_{\mathrm{TT}} = \sqrt{h}\,p_{\mathrm{TT}}$$

with $p_{\mathrm{TT}} \in S^2_{\mathrm{TT}}(T\Sigma, h)$. Let $h(q^\alpha) = \sigma(q^\alpha)$, then

$$\frac{\partial h_{ab}}{\partial q^\alpha} = \sigma_* \frac{\partial}{\partial q^\alpha}.$$

We may now introduce coordinates in the fiber of $T^*\mathcal{T}(\Sigma)$ by

$$p^\alpha = \int_\Sigma p_{\mathrm{TT}}^{ab} \frac{\partial h_{ab}}{\partial q^\alpha} \sqrt{h}\, \mathrm{d}^2 x, \tag{32}$$

see [4].

Conversely, given $(q, p) \in T^*\mathcal{T}(\sigma)$, with $h = \sigma(q)$, we can define p_{TT} by the conditions

$$p^\alpha = \int_\Sigma p_{\mathrm{TT}}^{ab} \frac{\partial h_{ab}}{\partial q^\alpha} \sqrt{h}\, \mathrm{d}^2 x, \quad \alpha = 1, \ldots, 6\,\mathrm{genus}(\Sigma) - 6.$$

Solve for λ using (31) and define π_{TT} by (30). Finally, (g, π) are defined by $g = \mathrm{e}^{2\lambda}h$ and π is given by (29). Now we put

$$\mathcal{Y}_{\tau,\sigma}(q, p) = (g, \pi). \tag{33}$$

4.2. Estimates for the evolution problem

Let $(g(\tau), \pi(\tau))$ be a solution to the evolution problem in CMC gauge, (13) and (14) starting at τ_0, with lapse satisfying (15) and with shift vectorfield X chosen so that the conformal metric $h(\tau) \in \sigma$ for all τ. Here $h(\tau)$ is the unique element in $\mathcal{M}_{-1}$ in the $\mathcal{P}$-orbit of $g(\tau)$. See [4] for a discussion of the choice of X. The shift vector field is chosen as the solution of an elliptic system with coefficients in h and is therefore estimated in terms of h.

If $\Lambda > 0$, we assume $\tau_0 > \sqrt{2\Lambda}$. By standard theory we have short time existence for this system, in particular there exist $\tau_{\min} < \tau_0 < \tau_{\max}$ so that we have solution for $\tau_{\min} < \tau < \tau_{\max}$.

Lemma 9 (Area estimate). *Let τ_0, $(g(\tau), \pi(\tau))$, $\tau_{\min}$, $\tau_{\max}$ be as above. The area of Σ w.r.t. g is given by*

$$\mathrm{Area}(\Sigma, g) = \int_\Sigma \sqrt{g}\, \mathrm{d}^2x. \tag{34}$$

Then for $\tau \in (\tau_{\min}, \tau_{\max})$

$$\frac{\tau_0^2 - 2\Lambda}{\tau^2 - 2\Lambda} \le \frac{\mathrm{Area}(\Sigma, g(\tau))}{\mathrm{Area}(\Sigma, g(\tau_0))} \le \left(\frac{\tau_0^2 - 2\Lambda}{\tau^2 - 2\Lambda}\right)^{1/2} \quad \text{if } \tau > \tau_0, \tag{35}$$

$$\left(\frac{\tau_0^2 - 2\Lambda}{\tau^2 - 2\Lambda}\right)^{1/2} \le \frac{\mathrm{Area}(\Sigma, g(\tau))}{\mathrm{Area}(\Sigma, g(\tau_0))} \le \frac{\tau_0^2 - 2\Lambda}{\tau^2 - 2\Lambda}, \quad \text{if } \tau < \tau_0. \tag{36}$$

Proof. We compute $\partial_\tau \mathrm{Area}(\Sigma, g)$. By assumption we are using CMC gauge, so $\mathrm{tr}_g \pi = \tau$. We compute

$$\begin{aligned}\partial_\tau \sqrt{g} &= \tfrac{1}{2}\mathrm{tr}_g(\partial_\tau g)\sqrt{g}\\ &= \tfrac{1}{2}\mathrm{tr}_g(-2N(g\,\mathrm{tr}_g\pi - \pi) + \mathcal{L}_X g)\sqrt{g}\\ &= (-N\tau + \mathrm{div}_g X)\sqrt{g}.\end{aligned}$$

Since Σ is compact,

$$\int_\Sigma \mathrm{div}_g X \sqrt{g}\, \mathrm{d}^2x = 0$$

and hence

$$\partial_\tau \mathrm{Area}(\Sigma, g) = -\int_\Sigma N\tau\sqrt{g}\, \mathrm{d}^2x,$$

which using inequality (22) for the lapse N gives the differential inequality for the area

$$-\frac{2\tau}{\tau^2 - 2\Lambda} \le \partial_\tau \log(\mathrm{Area}(\Sigma, g)) \le -\frac{\tau}{\tau^2 - 2\Lambda}.$$

Solving this inequality gives the result. □

Lemma 10 (Energy estimate). *Let* τ_0, $(g(\tau), \pi(\tau))$, $\tau_{\min}$, $\tau_{\max}$ *be as above.*

$$\widehat{E}(g(\tau)) \le \widehat{E}(g(\tau_0)) \left(\frac{\tau_0 + \sqrt{\tau_0^2 - 2\Lambda}}{\tau + \sqrt{\tau^2 - 2\Lambda}} \right)^{\sqrt{2}} \quad \text{if } \tau < \tau_0, \tag{37}$$

$$\widehat{E}(g(\tau)) \le \widehat{E}(g(\tau_0)) \left(\frac{\tau + \sqrt{\tau^2 - 2\Lambda}}{\tau_0 + \sqrt{\tau_0^2 - 2\Lambda}} \right)^{\sqrt{2}} \quad \text{if } \tau > \tau_0. \tag{38}$$

Proof. By the diffeomorphism invariance of $\widehat{E}$, $D\widehat{E}(g)\mathcal{L}_X g = 0$ and thus we have by (7), noting that the trace free part of $-2N(\mathrm{tr}_g \pi - \pi)$ is just $2N\pi_{\mathrm{TT}}$,

$$\partial_\tau \widehat{E}(g) = \int_\Sigma N \langle \pi_{\mathrm{TT}}^\sharp \nabla S^l, \nabla S^l \rangle_g \sqrt{g}\, \mathrm{d}^2 x,$$

which gives the estimate

$$|\partial_\tau \widehat{E}(g)| \le \max(N|\pi_{\mathrm{TT}}|_g)\widehat{E}(g).$$

Lemma 5 gives pointwise bounds on N and π_{TT} and we get the following differential inequality for $\widehat{E}$:

$$|\partial_\tau \widehat{E}(g(\tau))| \le \frac{\sqrt{2}}{\sqrt{\tau^2 - 2\Lambda}} \widehat{E}(g(\tau)).$$

Integrating this differential inequality completes the proof of the Lemma. □

4.3. *Proof of Theorem* 6

The reduction of $2+1$ gravity in the CMC gauge using the slice σ yields a smooth time-dependent Hamiltonian system on $T^*\mathcal{T}(\Sigma)$. First we prove that this has global existence in time. Then we reconstruct the solution curve in $T^*\mathcal{M}$, thus proving global existence for the original system. Since we are considering a smooth finite-dimensional time-dependent Hamiltonian system, we need only prove that the data do not blow up for τ satisfying the conditions of (i).

Let τ_0 satisfying the conditions of (i) be given and let $(q(\tau), p(\tau))$ be a solution curve with $(q(\tau_0), p(\tau_0)) = (q_0, p_0)$ and let $(g(\tau), \pi(\tau)) = \mathcal{Y}_\tau(q(\tau), p(\tau))$ be the corresponding solution curve in $T^*\mathcal{M}$. From standard theory it follows that we have existence for some interval $(\tau_{\min}, \tau_{\max})$ containing τ_0.

From Lemma 10 we know that the Dirichlet energy $\widetilde{E}(q)$ is bounded for $\tau \in (\tau_{\min}, \tau_{\max})$. By Proposition 2, the Dirichlet energy is a proper function on $\mathcal{T}(\Sigma)$ and therefore for such τ, $q(\tau)$ stays in a compact subset of $\mathcal{T}(\Sigma)$.

Now consider $h(\tau) = \sigma(q(\tau))$. By the construction of the slice σ we have under the above conditions on τ uniform pointwise estimates for $h(\tau)$. In the following we suppress reference to τ.

Let $g = e^{2\lambda}h$ with $h \in \mathcal{M}_{-1}$. Then by the considerations in Section 4.1 we have $\pi^{ab}_{\mathrm{TT}} = e^{-4\lambda}p^{ab}_{\mathrm{TT}}$. We estimate the coordinates in the fiber of $T^*\mathcal{T}(\Sigma)$ using (32) and the Hölder inequality,

$$\begin{aligned}
|p^\alpha| &= \left| \int_\Sigma p^{ab}_{\mathrm{TT}} \frac{\partial h_{ab}}{\partial q^\alpha} \sqrt{h}\, \mathrm{d}^2x \right| \\
&= \left| \int e^{-\lambda} p^{ab}_{TT} \frac{\partial h_{ab}}{\partial q^\alpha} e^{\lambda} \sqrt{h}\, \mathrm{d}^2x \right| \\
&\le \left\| \frac{\partial h}{\partial q^\alpha} \right\|_{h,\max} \left(\int_\Sigma e^{2\lambda} \sqrt{h}\, \mathrm{d}^2x \right)^{1/2} \left(\int_\Sigma e^{-2\lambda} p^{ab}_{\mathrm{TT}} p^{cd}_{\mathrm{TT}} h_{ac} h_{bd} \sqrt{h}\, \mathrm{d}^2x \right)^{1/2},
\end{aligned}$$

where $\|\partial h/\partial q^\alpha\|_{h,\max}$ is the maximum modulus w.r.t. h of the tensor $\partial h/\partial q^\alpha$. By construction this is uniformly bounded for τ satisfying our assumptions. Further $e^{2\lambda}\sqrt{h} = \sqrt{g}$ so

$$\int_\Sigma e^{2\lambda} \sqrt{h}\, \mathrm{d}^2x = \mathrm{Area}(\Sigma, g),$$

which is estimated in Lemma 9. It remains to consider the last factor. Note that, by the conformal transformation rules,

$$e^{-2\lambda} p^{ab}_{\mathrm{TT}} p^{cd}_{\mathrm{TT}} h_{ac} h_{bd} \sqrt{h} = \pi^{ab}_{\mathrm{TT}} \pi^{cd}_{\mathrm{TT}} g_{ac} g_{bd} \sqrt{g} = |\pi_{\mathrm{TT}}|^2_g \sqrt{g}.$$

Using (21) we now have the bound

$$|p^\alpha| \le \left\| \frac{\partial h}{\partial q^\alpha} \right\|_{h,\max} \mathrm{Area}(\Sigma, g) \left(\frac{\tau^2 - 2\Lambda}{2} \right)^{1/2}.$$

Referring again to the area estimate in Lemma 9 we find that the coordinates p^α in the fiber of $T^*\mathcal{T}(\Sigma)$ are bounded uniformly under the conditions on τ. Therefore it follows that we can extend the interval of existence for the reduced system to the intervals claimed in (i).

We have now proved global existence for the reduced version of 2 + 1 vacuum GR. It remains to reconstruct the solution curve. This is done using the map $\mathcal{Y}_{\tau,\sigma}$. By construction this is a smooth map. The lapse and shift are governed by elliptic equations which satisfy uniform estimates for τ satisfying the present assumptions. This finishes the proof of (i).

Point (ii) of Theorem 6 follows from Lemma 9 and finally point (iii) of Theorem 6 is easily verified.

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